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Merging Quantum Annealing Computation and Particle Statistics: A Prospect in the Search of Efficient Solutions to Intractable Problems

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In principle, a quantum Boolean network can be obtained by implementing gates as local relaxation processes (a quantum transposition of simulated annealing), while the correlations between Boolean variables imposed by gate wiring could be provided by particle statistics. It is argued that such a wiring should not affect the relaxation time of individual gates.

Testing different computation paradigms while searching for the best match with quantum specialties seems a natural thing to do. Quantum annealing computation A—the quantum transposition of classical "simulated annealing"—is one alternative to the most common approach of quantum sequential computation S, namely the quantum transposition of the sequential Turing machine (Deutsch, 1985; Ekert and Jozsa, 1996). Let us compare A and S. S relies on performing a sequence of reversible transformations onto a digital quantum register which can dwell in a coherent superposition of computational states (Fig. 1a, ignore dotted lines). To improve on classical efficiency, these states need to interfere with each other through some transformation. S has produced outstanding results, like factoring in polynomial time (Shor, 1994). However, the following difficulties still exist: (i) there is no "mechanical way" of applying S to improve on classical computation—finding new ways requires much ingenuity and is proving difficult; (ii) S is suspected to be unable to solve NP-complete problems in polynomial time (Bennet *et al.* 1994); (iii) S is severely hampered by decoherence (Harroche and Raimond, 1996).

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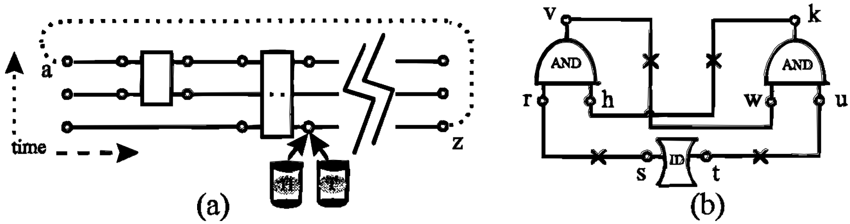

Fig. 1.

We argue that "difficulties" (i) and (ii) might derive from the *sequential character* of the physical computation process (Feynman, 1986: "sequentiality is a logical, not a physical requirement"). The *time diagram* of Fig. 1a shows a Boolean network of N nodes, or qubits, labeled a to z—although the register is three qubits. This network is satisfiable by any input, owing to its constrained topology. Connecting a to z (dotted line) is logically legitimate and makes the state of z participate in logically determining itself through a loop of logical implications. Consequences are: (1) testing network satisfiability is now an NP-complete problem, and (2) the physical process should undergo a closed timelike line (an implication loop becomes a causal loop). Thus, a possible problem with S is its inability to map a network of unconstrained topology.

It is not so with A. We shall review classical annealing first (Fig. 1a, ignore dashed lines). Now all nodes a to z exist at the same time. They can be seen as glasses containing a coin which can either be in the head (0) or tail (1) state. Computation is: (1) Do coin states satisfy all *local* gate and wire relations? (2) If yes, stop: a solution has been found. (3) If not, shake (flip coins) and go to (1). If this process does not stop, after n loops one can decide with any confidence level desired (increasing with n) that the network is not satisfiable. Wire a–z does not bar this kind of computation.

To speed up computation, each gate or wire (network element—N.E.) is submitted to a local energy function. The energies of the *local* coin configurations which satisfy the N.E. are the same and correspond to a degenerate ground state. The energies of the others are discretely above ground. "Shaking" is now associated with a gradient driving the process to the ground state. But there can be frustration between different network parts which have separately reached ground. The overall energy has many *local* minima, of the order of 2^N.

In quantum transposition, each node r becomes a qubit of eigenstates $|\chi\rangle_r$, with $\chi = 0, 1$. We will refer to Fig. 1b, where (1) all wires incorporate a NOT function (denoted by X) and will be called links, (2) logically irreversible gates can be used (inputs and outputs coexist), (3) ID stands for identity gate, (4) nodes are a set of interlinked pairs. This closed network is universal and can wire up any Boolean constant, thus preimposed inputs or outputs.

The Hamiltonian of an individual AND gate, operating on qubits r, h, v, has eight eigenstates, all the combinations of qubit eigenstates. Eigenstates $|0\rangle_r|0\rangle_h|0\rangle_u$, . . ., $|1\rangle_r|1\rangle_h|1\rangle_v$, which satisfy the gate, belong to a fourfold degenerate ground state. Conversely, $|0\rangle_r|0\rangle_h|1\rangle_v$, . . .,$|1\rangle_r|1\rangle_h|0\rangle_v$ stand on a discretely higher (say by ΔE) energy level. The ID gate is dealt with similarly. Hamiltonians of such gates are given in Castagnoli and Rasetti (1993). By considering a gate alone, annealing means bringing it to relax onto ground by lowering the temperature of the heat bath. Even at fixed temperature T such that $0 < kT << \Delta E$, the probability p of finding the gate relaxed by time t will increase with time following (asymptotically) the law $p = 1 - e^{-\rho t}$, where $\rho > 0$ (incidentally, even without a heat bath, tunneling would bring the state to ground).

Correlations between Boolean variables established by links will rely on particle statistics. To this purpose, the qubit representation will be viewed as the outcome of a "finer" representation. Label l and eigenvalue χ of qubit $|\chi\rangle_l$ will become two binary *compatible* attributes of a fermion particle, χ ranging over 0, 1 and l ranging over the node labels of a link. Attribute compatibility will play an essential role: in principle, one should be able to alter the state of one attribute without affecting the state of the other. By way of exemplification, χ can be interpreted as the *spin* of a spin-1/2 particle with respect to some axis (the same for all qubits), l the label of the *spatial state* occupied by the particle, namely a *site* of some lattice L—network nodes will be viewed as the sites of L. Here *space* and *spin* are the compatible attributes, but there might be other interpretations.

Each site should contain one fermion exactly. To create L, an independent Hamiltonian H_{rs} is introduced for each link (r, s) (gate Hamiltonians are "turned off" for the time being). H_{rs} operates on the *spatial state* of a couple of identical fermions 1 and 2 through kinetic and Coulomb potential (external and interaction) operators (1 and 2 have to be charged particles), while it does not contain spin operators. The discrete Hilbert space of the labels of the sites of the two particles is span$\{|r\rangle_1|r\rangle_2, |r\rangle_1|s\rangle_2, |s\rangle_1|r\rangle_2, |s\rangle_1|s\rangle_2\}$, where $|r\rangle_1$ reads particle 1 in label r site, etc. For example the eigenvalues-eigenstates of

$$H_{rs} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \text{are} \quad \begin{array}{ll} 4\text{—} & |s\rangle_1|s\rangle_2 \\ 3\text{—} & |r\rangle_1|r\rangle_2 \\ 2\text{—} & \frac{1}{\sqrt{2}}(|r\rangle_1|s\rangle_2 - |s\rangle_1|r\rangle_2) \\ 0\text{—} & \frac{1}{\sqrt{2}}(|r\rangle_1|s\rangle_2 + |s\rangle_1|r\rangle_2) \end{array}$$

When all links are in their ground states, each link site (network node) hosts one particle exactly. Moreover, the spatial state vector of the two particles belonging to each link is symmetrical under their permutation P_{12}. Thus, the spin state vector of the two particles (factorizable, since H_{rs} has no spin operators) is atisymmetrical. The overall link ground state is

$$|\psi\rangle = \frac{1}{2}(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2)(|r\rangle_1|s\rangle_2 + |s\rangle_1|r\rangle_2) \tag{1}$$

where $|0\rangle_1$ reads spin of particle 1 zero (down), etc.

Thus

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\psi'\rangle - |\psi''\rangle) = \frac{1}{\sqrt{2}}(|0\rangle_r|1\rangle_s - |1\rangle_r|0\rangle_s)$$

where

$$|\psi'\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|r\rangle_1|1\rangle_2|s\rangle_2 - |1\rangle_1|s\rangle_1|0\rangle_2|r\rangle_2) = \eta'|0\rangle_r|1\rangle_s$$

$$|\psi''\rangle = \frac{1}{\sqrt{2}}(|1\rangle_1|r\rangle_1|0\rangle_2|s\rangle_2 - |0\rangle_1|s\rangle_1|1\rangle_2|r\rangle_2) = \eta''|1\rangle_r|0\rangle_s$$

Notice that both terms appearing in $|\psi'\rangle$ ($|\psi''\rangle$) in the particle representation map onto $|0\rangle_r|1\rangle_s$ ($|1\rangle_r|0\rangle_s$) in the qubit representation. η' and η'' are phase factors which must be equal for $|\psi\rangle$ to remain invariant for a rotation of the spin reference axis.

Let $A_{12} = 1 - P_{12}$ plus normalization be the antisymmetric operator. Notice that also $A_{12}|\psi'\rangle = |\psi'\rangle$ and $A_{12}|\psi''\rangle = |\psi''\rangle$. The $|\psi\rangle$ antisymmetry (even under permutation of site labels) is "oversized" and can be broken to the normal antisymmetry (under P_{12}) of $|\psi'\rangle$ and $|\psi''\rangle$. Furthermore,

$$A_{12}|\chi\rangle_1|r\rangle_1|\chi\rangle_2|s\rangle_2 = A_{12}|\chi\rangle_1|s\rangle_1|\chi\rangle_2|r\rangle_2 = 0 \qquad \text{for} \quad \chi = 0, 1$$

This, in the qubit representation, becomes symmetry A_{rs} defined by

$$A_{rs}|\chi\rangle_r|1-\chi\rangle_s = |\chi\rangle_r|1-\chi\rangle_s, \qquad A_{rs}|\chi\rangle_r|\chi\rangle_s = 0, \qquad \text{with} \quad \chi = 0, 1$$

Thus, having turned link Hamiltonians on and allowed the links to relax onto ground (independently of each other), the lattice of "qubits" L is obtained, where qubit eigenvalues are submitted to the symmetry (concerning pairs of qubits connected by a link) $\tilde{A} = \prod_{r,s} A_{rs}$, where r, s range over all pairs of link labels.

The spin of the particle in a link site can be measured *in principle* without disrupting L and $\tilde{A}$, the result of dynamics on spatial attributes *compatible* with spin. $|\psi\rangle$ is projected onto either $|\psi'\rangle$ or $|\psi''\rangle$. Since $\tilde{A}$

symmetry is kept throughout the measurement, these projections can be written $A_{rs}\, P'\, |\psi\rangle = A_{rs}|\psi\rangle' = |\psi\rangle'$, $A_{rs}P''|\,\psi\rangle = A_{rs}|\psi\rangle'' = |\psi\rangle''$, where $P' = |\psi'\rangle\,\langle\psi|$, etc.

After creating L and $\tilde{A}$, gate Hamiltonians are "turned on," then gates are allowed to relax, both things operating on the compatible spin attribute without disrupting L and $\tilde{A}$. Let P_g be the projector representing relaxation of gate # g, and $|\Psi\rangle = \Sigma_h \alpha_h |\Psi_h\rangle$ the generic network state, where $|\Psi_h\rangle$ is a tensor product of of all qubit eigenstates. At relaxation, $|\Psi\rangle$ satisfies

$$\prod_{r,s} A_{rs} \prod_{g} P_g |\Psi\rangle = |\Psi\rangle \tag{2}$$

where g runs over all gate labels. Any eigenstate $|\Psi_h\rangle$ satisfying equation (2) satisfies all links and gates, and is thus a network solution (the case without solutions will be discussed below). Notice that permutations between fermions belonging to different links would not establish any spin correlation even if fermions were identical across links, since link Hamiltonians are independent of each other. Entanglement with a reservoir initially in state $|R\rangle$ changes each $|\Psi_h\rangle|R\rangle$ into $|\Psi_h\rangle|R_h\rangle$, where $|R_h\rangle$ is now correlated with $|\Psi_h\rangle$, without altering the result of equation (2). A is thus unaffected by decoherence, as long as link ground states are not offset by external interaction.

We shall now discuss relaxation time under symmetry $\tilde{A}$. This appears to be a basic question raised by the present approach. If $\tilde{A}$ (particle statistics) were a classical bound, this would likely be exponential in N. Here we hypothesize that $\tilde{A}$ works on the state originated by the independent gate relaxation processes like a quantum watchdog, continuously killing at its origin—through a series of "infinitesimal" state vector reductions—any path that would lead to violating $\tilde{A}$ itself. $\tilde{A}$ would work exactly as in the antisymmetrization operation $\tilde{A}|\Psi\rangle = |\Psi\rangle$. If $|\Psi\rangle$ is out of symmetry, it is projected through destructive and constructive interference onto the subspace $H_{\tilde{A}}$ of the states satisfying $\tilde{A}$. This can be seen as continuous partial reduction of $|\Psi\rangle$. Thus relaxation time would be that of a population of network elements relaxing independently of each other (first links, then gates). Since relaxation of a N.E. follows a law of the type $p(t) = 1 - e^{-\rho t}$, it can be seen that the time τ required to reach any desired probability q that all N.E. have relaxed is polynomial in their number, or N. Measurement of the network qubits at time τ yields a solution with probability q (disregarding lucky chances). That this is a solution is checkable off line in polynomial time. If, instead, the network has no solutions, that can be decided after a number of repetitions of the overall process of preparation and measurement, which, for a given confidence level, grows polynomially with N.

We shall now discuss the results obtained. Quantum annealing relies on physical principles and it is not yet clear how to implement it. If implementable, there might be benefits. We believe it constitutes a potentially interesting prospect. We conjecture that an evolution resulting from continuous reduction of a relaxation process onto $H_{\tilde{A}}$ does not comply with the mechanistic notion of evolutions either determined from the past or partly freed from it by reduction—i.e., randomness. Our evolution appears to be (in a kind of elusive way) affected by the future, when it is prevented from undergoing an infinitesimal reduction whose "immediately future" outcome would violate $\tilde{A}$. $\tilde{A}$ would work like an *oracle* in computer science, preventing the process from choosing a path which would lead to a nonsolution. This would create faster than classical computation. Such a conjecture can be investigated within the model of reduction driven by both forward- and backward-in-time causality developed in Castagnoli (1995) and Castagnoli and Rasetti (1996).

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